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Engineering Notes

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Improved Method for Calculating Exact Geodetic Latitude and Altitude

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Introduction

THIS Engineering Note explicitly derives exact, singularity-free expressions for the geodetic latitude and altitude of an arbitrary point in space. Such a problem entails the solution of a quartic equation.[1–7] Some references pinpoint the unique solution from among the four solutions of the quartic. However, nearly all of those solutions contain singularities at either a pole, the equator, or both. In the single treatment claimed to be singularity free,[2] the derivation is not systematic or explicitly spelled out.

Formulation

As usual, assume the Earth to be an oblate spheroid, i.e., an ellipsoid of revolution whose semimajor axis a is the radius of the circle described by the equatorial plane and whose semiminor axis b is a line joining its center and a pole. Choose an Earth-fixed Cartesian coordinate system with the origin at the center of the Earth ellipsoid. The unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ are along the x, y, and z axes, respectively. The $+z$ axis points in the direction of the North Pole, the $+x$ axis is the line of intersection of the equatorial plane with the plane of zero longitude, and the $+y$ axis completes a right-handed coordinate system. An equation of the ellipsoid in this frame is $(r/a)^2 + (z/b)^2 = 1$, where $r = \sqrt{(x^2 + y^2)}$. Let $P(x_0, y_0, z_0)$ be the coordinates of the given point. It is desired to find the nearest point $P'(x, y, z)$ to P on the surface of the ellipsoid.

The slope of a normal to the ellipsoid at any point on its surface is given by

$$-\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^{-1} = \frac{a^2 z}{b^2 r}, \qquad \text{where} \qquad \left(\frac{r}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1 \tag{1}$$

Therefore, the slope of a normal from P is $(z_0 - z)/(r_0 - r) = a^2 z/(b^2 r)$, where $r_0 = \sqrt{(x_0^2 + y_0^2)}$, i.e., $r_0 a^2 z = r[(a^2 - b^2)z + b^2 z_0]$. Squaring both sides and expressing r in terms of z, $a^2 b^2 r_0^2 z^2 = (b^2 - z^2)[(a^2 - b^2)z + b^2 z_0]^2$. Writing this equation in descending powers of z,

$$(a^2 - b^2)^2 z^4 + 2b^2(a^2 - b^2)z_0 z^3 + b^2\Big[a^2 r_0^2 + b^2 z_0^2 - (a^2 - b^2)^2\Big]z^2 - 2b^4(a^2 - b^2)z_0 z - b^6 z_0^2 = 0$$

Because z is either positive or negative and sign z = sign z_0, this quartic can be expressed in terms of $|z|$. The result is

$$|z|^4 + 2p|z|^3 + q|z|^2 - 2pb^2|z| - p^2 b^2 = 0 \tag{2}$$

where $p = |z_0|/e'^2$, $q = p^2 - b^2 + r_0^2/(e^2 e'^2)$, $e^2 = 1 - (b/a)^2$, and $e'^2 = (a/b)^2 - 1$.

Two of three powerful theorems[8] will now be invoked to expose the nature of the roots of Eq. (2). The third theorem will be needed later.

1) An equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$ and cannot have more negative roots than there are changes of sign in $f(-x)$.

2) Every equation that is of an even degree and has its last term negative has at least two real roots, one positive and one negative.

3) Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term.

From Eq. (2) and the fact that p is positive, the following conclusions are deduced. 1) Because the last term of Eq. (2) is negative, there are at least two real roots of opposite sign (by Theorem 2); and 2) because there is only one change of sign in Eq. (2), there is at most one positive root (by Theorem 1). From conclusions 1 and 2, it is seen immediately that Eq. (2) has exactly one positive root, and this is the root that is sought.

The solution of Eq. (2) is effected by a standard method known as Ferrari's method.[8] In Eq. (2), add to each side $(c|z| + d)^2$, the quantities c and d being determined so as to make the left-hand side a perfect square; then

$$|z|^4 + 2p|z|^3 + (q + c^2)|z|^2 + 2(cd - pb^2)|z| + d^2 - p^2 b^2 = (c|z| + d)^2$$

Suppose that the left-hand side of the equation equals $(|z|^2 + p|z| + t)^2$; then comparing the coefficients,

$$p^2 + 2t = q + c^2, \qquad pt = cd - pb^2, \qquad t^2 = d^2 - p^2 b^2 \tag{3}$$

Eliminating c and d from these equations, $p^2(t + b^2)^2 = (2t + p^2 - q)(t^2 + p^2 b^2)$, or

$$2t^3 - qt^2 - \frac{a^2 r_0^2 z_0^2}{e'^8} = 0 \tag{4}$$

Applying Theorems 1 and 3 to Eq. (4), we see that Eq. (4) has exactly one positive root. The solution of Eq. (4) is accomplished by a standard method known as Cardan's solution.[8] Eliminating the t^2 term in Eq. (4) by making the substitution $t = t' + q/6$, we obtain

$$t'^3 + Ft' + H = 0, \quad \text{where} \quad F = -\frac{q^2}{12}, \qquad H = -\frac{q^3}{108} - \frac{a^2 r_0^2 z_0^2}{2e'^8} \tag{5}$$

The solution of Eq. (5) is

$$t' = \left(\sqrt{H^2/4 + F^3/27} - H/2\right)^{\frac{1}{3}} - \left(\sqrt{H^2/4 + F^3/27} + H/2\right)^{\frac{1}{3}} \tag{6}$$

Equation (6) is valid provided that $H^2/4 + F^3/27 \geq 0$. We have $H = -2(q^3 + 2P)/6^3$, where $P = 27a^2 r_0^2 z_0^2/e'^8$. Thus, $H^2/4 + F^3/27 = 4P(P + q^3)/6^6$. Therefore, if $H^2/4 + F^3/27 \geq 0$, then $P + q^3 \geq 0$, which is satisfied when $q \geq 0$, i.e., when $a^2 r_0^2 + b^2 z_0^2 \geq (a^2 - b^2)^2$, which represents all space excluding the open region bounded by the ellipsoid $[r_0/(ae^2)]^2 + [z_0/(be'^2)]^2 = 1$. Because $ae^2 < be'^2 < 43$

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km, this region is of no practical interest. Therefore, $q \geq 0$ will be our constraint.

Applying Theorem 1 to Eq. (4) with $-t$ substituted for t, we see that Eq. (4) has no negative roots. Hence, Eq. (4) has only one real root and it is positive.

Therefore, $[\sqrt{(H^2/4 + F^3/27)} - H/2]^{1/3} = Q/6$, where $Q = [\sqrt{(P+q^3)} + \sqrt{P}]^{2/3}$. Similarly, $[\sqrt{(H^2/4 + F^3/27)} + H/2]^{1/3} = -Q'/6$, where $Q' = [\sqrt{(P+q^3)} - \sqrt{P}]^{2/3}$. (Note that $Q' = q^2/Q$. However, putting Q' in this form is undesirable as it introduces a singularity when $Q = 0$, which occurs when either $r_0 = 0$ and $|z_0| = be'^2$ or $r_0 = ae^2$ and $z_0 = 0$.) Substituting these values into Eq. (6), $t' = (Q + Q')/6 \Rightarrow t = (q + Q + Q')/6$.

It will now be shown that $q \leq 2t \leq q + b^2$, where the left-hand side equality occurs when either r_0 or $z_0 = 0$ and the right-hand-side equality occurs when $ar_0 = b|z_0|$. Rewriting Eq. (4) in the form $t^2(2t - q) = a^2 r_0^2 z_0^2 / e'^8$, the lower bound of t is deduced at once. Now let $g = \upsilon^3 - q\upsilon^2 - 4a^2 r_0^2 z_0^2/e'^8$, where $\upsilon \geq q$. Then, $dg/d\upsilon = \upsilon(3\upsilon - 2q) \geq 0$. Thus, g is a monotonically increasing function of υ for all $\upsilon \geq q$. Because $2t \geq q$ and $g = 0$ when $\upsilon = 2t$, it follows that $g \geq 0$ when $\upsilon \geq 2t$. Putting $\upsilon = q + b^2$ in the expression for g, we obtain

$$g|_{\upsilon = q + b^2} = b^2\left(p^2 - \frac{r_0^2}{e^2 e'^2}\right)^2 \geq 0 \Rightarrow 2t \leq q + b^2$$

equality occurring when $p = r_0/(ee')$, i.e., when $r_0/e = |z_0|/e'$ or when $ar_0 = b|z_0|$, in which case $q = 2p^2 - b^2$ and $t = (q + b^2)/2 = p^2$.

Solving for c and d from the system of equations (3), we obtain $c = \sqrt{(p^2 - q + 2t)}$ and $d = \sqrt{(t^2 + p^2 b^2)}$. Because $2t \geq q$, it follows that $c \geq p$. Now, $(|z|^2 + p|z| + t)^2 = (c|z| + d)^2 \Rightarrow |z|^2 + p|z| + t = \pm(c|z| + d)$, from which the following two quadratics in $|z|$ are obtained:

$$|z|^2 + (p - c)|z| + t - d = 0 \qquad (7)$$

and

$$|z|^2 + (p + c)|z| + t + d = 0 \qquad (8)$$

It has already been shown that Eq. (2) has exactly one positive root. Furthermore, because $t^2 - d^2 = -p^2 b^2 \leq 0$ and $d \geq 0$, it follows that $t - d \leq 0 \leq t + d$. Now, $t - d$ is the last term of Eq. (7) and has just been shown to be negative. Hence, by applying Theorem 2 to Eq. (7), we see immediately that Eq. (7) must contain the required positive root. Hence,

$$|z| = \frac{-(p - c) + \sqrt{(p - c)^2 - 4(t - d)}}{2}$$

$$= \frac{c - p + \sqrt{2p^2 - q - 2t - 2pc + 4\sqrt{t^2 + p^2 b^2}}}{2}$$

This solution is free of singularities. It is easy to show that

$$\lim_{z_0 \to 0} |z| = 0 \qquad \text{and} \qquad \lim_{r_0 \to 0} |z| = b$$

Proceeding with our derivation, $r = a\sqrt{[1 - (z/b)^2]}$, $x = x_0 r/r_0$, $y = y_0 r/r_0$, and $\lambda = \arctan(y_0/x_0)$, where $r_0 \neq 0$. [Here, λ is the longitude at (x, y).]

To compute the geodetic latitude and altitude at $P'(x, y, z)$, it is desirable to introduce a geometric term N_e, which is never zero. N_e is defined to be the distance along the ellipsoidal normal from the surface of the ellipsoid to the z axis (Fig. 1).

From Fig. 1, $\cos\chi = r/N_e$. Also, $\tan\chi = a^2 z/(b^2 r)$ [from Eq. (1)] $\Rightarrow N_e = r\sec\chi = a\sqrt{(1 + e'^2 z^2/b^2)}$, $\sin\chi = \cos\chi\tan\chi = a^2 z/(b^2 N_e) \Rightarrow \chi = \arcsin[a^2 z/(b^2 N_e)]$, which is the desired expression for the geodetic latitude. Also from Fig. 1, $h\cos\chi = r_0 - r = r_0$

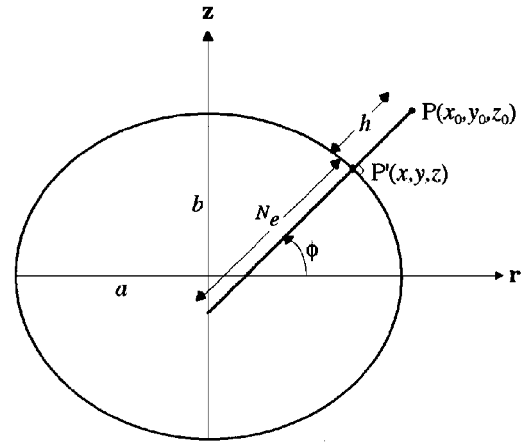

Fig. 1 Ellipsoidal normal.

$-N_e\cos\chi$, $h\sin\chi = z_0 - z = z_0 - (b/a)^2 N_e \sin\chi$. Multiplying the first equation by $\cos\chi$ and the second by $\sin\chi$ and adding,

$$h = r_0\cos\chi + z_0\sin\theta - N_e[\cos^2\chi + (b/a)^2\sin^2\chi]$$

$$= r_0\cos\chi + z_0\sin\chi - a^2/N_e$$

which is the desired expression for the geodetic altitude.

To recapitulate, given a, b, x_0, y_0, and z_0, the algorithm is as follows:

$$e^2 = 1 - \left(\frac{b}{a}\right)^2, \qquad e'^2 = \left(\frac{a}{b}\right)^2 - 1$$

$$r_0 = \sqrt{x_0^2 + y_0^2}, \qquad p = \frac{|z_0|}{e'^2}, \qquad q = p^2 - b^2 + \frac{r_0^2}{e^2 e'^2}$$

If $q \geq 0$, then

$$P = \frac{27a^2 r_0^2 z_0^2}{e'^8}, \qquad Q = \left(\sqrt{P + q^3} + \sqrt{P}\right)^{\frac{2}{3}},$$

$$Q' = \left(\sqrt{P + q^3} - \sqrt{P}\right)^{\frac{2}{3}}, \qquad t = \frac{q + Q + Q'}{6}$$

$$c = \sqrt{p^2 - q + 2t}$$

$$z = \text{sign}(z_0)\frac{c - p + \sqrt{2p^2 - q - 2t - 2pc + 4\sqrt{t^2 + p^2 b^2}}}{2}$$

$$N_e = a\sqrt{1 + \frac{e'^2 z^2}{b^2}}, \qquad \chi = \arcsin\left(\frac{a^2 z}{b^2 N_e}\right)$$

$$h = r_0\cos\chi + z_0\sin\chi - \frac{a^2}{N_e}$$

Conclusion

Simple, unique, and exact expressions for the geodetic latitude and altitude of an arbitrary point have been derived. The point in question can be anywhere in space, excluding a small open region bounded by a prolate spheroid that is concentric with and contained within the Earth ellipsoid. The derivation is systematic, and the resulting expressions are free of singularities. The treatment presented is quite elegant and, to our knowledge, does not exist in the literature.

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Controller Reduction Using Normal Coordinates of Reconstruction Error Matrix and Component Cost Analysis Method

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Introduction

CONTROLLER design using linear quadratic Gaussian (LQG) procedures yields compensators of the same order as the plant. Many physical systems, such as aeroservoelastic systems, involve a large number of states and, therefore, result in high-order controllers. These high-order controllers are difficult to implement and may be susceptible to reliability problems. Therefore, it is essential to find ways to reduce the order of these controllers.

The elements of the component cost analysis method (CCA) were developed in Refs. 1–3. The method is based on the decomposition of the quadratic performance index of the plant-controller system into components that indicate the contributions of the different controller states to its optimal value. The states with negligible contributions to the quadratic performance index J are eliminated, thus reducing the order of the controller by truncation. The CCA method, however, is highly sensitive to the set of chosen coordinates, and one can readily show that it can lead to incorrect results.[4] The advantages of using observer normal coordinates in the process of controller reduction were pointed out in Ref. 4. However, because the CCA method could not be trusted while reducing the order of the controller, a brute force method[4] was suggested, by which all of the optimal regulator states, except for a single state, were used for feedback in the plant-regulator system, and J and eigenvalues for the closed-loop system were computed. This procedure was repeated, with a different single state omitted each time from the optimal regulator feedback expression, until all of the states had been scanned in such a manner, and the resulting effects on J and on the eigenvalues were computed. In the present Note, a modification is introduced in the observer equations,[4] which permits strict application of the principle of separation to the plant-reduced controller system. It then avoids the lengthy brute force method of identifying the states that can be neglected by adopting elements of the CCA method. The CCA method, as used in this Note, differs from the method developed in Refs. 1–3 on three main points. First, it is applied to the optimal regulator only, rather than the plant-controller system. Second, it uses normal coordinates of the reconstruction error matrix while performing component cost analysis in the optimal regulator's performance index. Third, which is related to the previous point, truncation is applied to diagonal matrices only. It will be shown that the described normal coordinates are essential for controller reduction based on CCA analysis. Thus, it can be seen that the method proposed combines basic elements of the controller reduction method developed in Ref. 4 with elements of the CCA method presented in Refs. 1–3.

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Optimal Linear Quadratic Regulator and the CCA Method

Let the system to be controlled be linear and time invariant, given in the following state-space form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{G}w \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{v} \tag{2}$$

where $\boldsymbol{A}$ is an $n \times n$ plant matrix, $\boldsymbol{B}$ is an $n \times l$ input matrix, $\boldsymbol{C}$ is an $m \times n$ output matrix, and v and w are zero mean uncorrelated white noise processes with intensities $\boldsymbol{V} > 0$ and $W \geq 0$. Let the steady-state performance index J be defined by

$$J = \lim_{t \to \infty} E(\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u}) \tag{3}$$

where $\boldsymbol{Q} \geq 0$ and $\boldsymbol{R} > 0$. Following optimal regulator theory, the input $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = -\boldsymbol{F}\boldsymbol{x} \tag{4}$$

where $\boldsymbol{F}$ is an $l \times n$ feedback gain matrix obtained through the solution of the usual algebraic Riccati equations. Substitution of Eq. (4) into Eq. (3) yields

$$J = \lim_{t \to \infty} E[\boldsymbol{x}^T(\boldsymbol{Q} + \boldsymbol{F}^T\boldsymbol{R}\boldsymbol{F})\boldsymbol{x}] \tag{5}$$

Following the CCA method, the cost ψ_i, which is associated with the x_i state, is defined by

$$\psi_i = \mathrm{tr}[\boldsymbol{x}\boldsymbol{x}^T(\boldsymbol{Q} + \boldsymbol{F}^T\boldsymbol{R}\boldsymbol{F})]_{ii} \tag{6}$$

so that

$$J = \sum_{i=1}^{n} \psi_i \tag{7}$$

Assume we now rank the regulator states so that

$$|\psi_1(x)| \geq |\psi_2(x)| \geq \cdots \geq |\psi_n(x)| \tag{8}$$

Then those n_u states with the smallest associated $|\psi|$ values are deleted, thus reducing the plant-regulator system, by truncation, to n_I important states, where $n_I = n - n_u$. The main difficulties associated with the CCA method can be traced to the truncation of the plant-regulator system. It may be argued that the regulator feedback gains associated with the n_u negligible $|\psi|$ values may, indeed, be set to zero with no appreciable effects on the value of the performance index J. However, it would be incorrect to proceed further and assume that the truncation of the plant, obtained by deleting those n_u unimportant states, will have little effect on the dynamics of the plant and, therefore, leave the values of the important states essentially unchanged. Therefore, in the remaining part of this work, the CCA method will be used only for the determination of the regulator feedback gains [in Eq. (4)] that can be set to zero without appreciable effects on the value of the performance index J.